Calculus, Advanced Placement

In this technological age, mathematics is more important than ever. When students leave school, they are more and more likely to use mathematics in their work and everyday lives — operating computer equipment, planning timelines and schedules, reading and interpreting data, comparing prices, managing personal finances, and completing other problem-solving tasks. What they learn in mathematics and how they learn it will provide an excellent preparation for a challenging and ever-changing future.

The state of Indiana has established the following mathematics standards to make clear to teachers, students, and parents what knowledge, understanding, and skills students should acquire in Calculus, Advanced Placement:

Standard 1 — Limits and Continuity

Students develop an understanding of the concept of limit by estimating limits from graphs and tables of values, and finding limits by substitution, and factoring rational functions. They extend the idea of a limit to one-sided limits and limits at infinity. They use limits to define and understand the concept of continuity, decide whether a function is continuous at a point, and find types of discontinuities. And they understand and apply two continuity theorems: the Intermediate Value Theorem and the Extreme Value Theorem.

Standard 2 — Differential Calculus

Students develop an understanding of the derivative as an instantaneous rate of change, using geometrical, numerical, and analytical methods. They use this definition to find derivatives of many types of functions and combinations of these functions (using, for example, sums, composites, and inverses). They also find second and higher order derivatives. They understand and use the relationship between differentiability and continuity. They understand and apply the Mean Value Theorem.

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Standard 3 — Applications of Derivatives

Students apply what they learn about derivatives to finding slopes of curves and the related tangent lines. They analyze and graph functions, finding where they are increasing or decreasing, their maximum and minimum points, their points of inflection, and their concavity. They solve optimization problems, find average and instantaneous rates of change (including velocities and accelerations), and model rates of change.

Standard 4 — Integral Calculus

Students understand that integration is used to find areas and they evaluate integrals using rectangular approximations. From this, they develop the idea that integration is the inverse operation to differentiation — the Fundamental Theorem of Calculus. They use this result to find definite and indefinite integrals, including using the method of integration by substitution. They also apply approximate methods, such as the Trapezoidal Rule, to find definite integrals.

Standard 5 — Applications of Integration

Students apply what they learn about integrals to finding velocities from accelerations, solving separable differential equations, and finding areas and volumes. They also apply integration to model and solve problems in physics, biology, economics, etc.

As part of their instruction and assessment, students should also develop the following learning skills by Grade 12 that are woven throughout the mathematics standards:

Mathematical Reasoning and Problem Solving

In a general sense, mathematics is problem solving. In all of their mathematics, students use problem-solving skills: they choose how to approach a problem, they explain their reasoning, and they check their results. At this level, students apply these skills to investigating limits and applying them to continuity, differentiability, and integration.

Communication

The ability to read, write, listen, ask questions, think, and communicate about math will develop and deepen students' understanding of mathematical concepts. Students should read text, data, tables, and graphs with comprehension and understanding. Their writing should be detailed and coherent, and they should use correct mathematical vocabulary. Students should write to explain answers, justify mathematical reasoning, and describe problem-solving strategies.

Representation

The language of mathematics is expressed in words, symbols, formulas, equations, graphs, and data displays. The concept of one-fourth may be described as a quarter, $\frac{1}{4}$, one divided by four, 0.25, $\frac{1}{8} + \frac{1}{8}$, 25 percent, or an appropriately shaded portion of a pie graph. Higher-level mathematics involves the use of more powerful representations: exponents, logarithms, π, unknowns, statistical representation, algebraic and geometric expressions. Mathematical operations are expressed as representations: $+$, $=$, divide, square. Representations are dynamic tools for solving problems and communicating and expressing mathematical ideas and concepts.

Connections

Connecting mathematical concepts includes linking new ideas to related ideas learned previously, helping students to see mathematics as a unified body of knowledge whose concepts build upon each other. Major emphasis should be given to ideas and concepts across mathematical content areas that help students see that mathematics is a web of closely connected ideas (algebra, geometry, the entire number system). Mathematics is also the common language of many other disciplines (science, technology, finance, social science, geography) and students should learn mathematical concepts used in those disciplines. Finally, students should connect their mathematical learning to appropriate real-world contexts.

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Standard 1

Limits and Continuity

Students understand the concept of limit, find limits of functions at points and at infinity, decide if a function is continuous, and use continuity theorems.

C.1.1 **Understand the concept of limit and estimate limits from graphs and tables of values.**

Example: Estimate $\lim\limits_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$ by calculating the function's values for $x = 2.1, 2.01, 2.001$ and for $x = 1.9, 1.99, 1.999$.

C.1.2 **Find limits by substitution.**

Example: Find $\lim\limits_{x \to 5} (2x + 1)$.

C.1.3 **Find limits of sums, differences, products, and quotients.**

Example: Find $\lim\limits_{x \to \pi} (\sin x \bullet \cos x + \tan x)$.

C.1.4 **Find limits of rational functions that are undefined at a point.**

Example: Find $\lim\limits_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$ by factoring and canceling.

C.1.5 **Find one-sided limits.**

Example: Find $\lim\limits_{x \to 4^-} \sqrt{4 - x}$.

C.1.6 **Find limits at infinity.**

Example: Find $\lim\limits_{x \to \infty} \frac{x}{x - 1}$.

C.1.7 **Decide when a limit is infinite and use limits involving infinity to describe asymptotic behavior.**

Example: Find $\lim\limits_{x \to 0} \frac{1}{x^2}$.

C.1.8 **Find special limits such as $\lim\limits_{x \to 0} \frac{\sin x}{x}$.**

Example: Use a diagram to show that the limit above is equal to 1.

C.1.9 **Understand continuity in terms of limits.**

Example: Show that $f(x) = 3x + 1$ is continuous at $x = 2$ by finding $\lim\limits_{x \to 2} (3x + 1)$ and comparing it with $f(2)$.

C.1.10 **Decide if a function is continuous at a point.**

Example: Show that $f(x) = \frac{x^2 + 2x - 8}{x - 2}$ is continuous at $x = 2$, provided that you define $f(2) = 6$.

C.1.11 **Find the types of discontinuities of a function.**

Example: What types of discontinuities has $h(x) = \frac{x^2 - 5x + 6}{x^2 - 4}$? Explain your answer.

C.1.12 **Understand and use the Intermediate Value Theorem on a function over a closed interval.**

Example: Show that $g(x) = 3 - x^2$ has a zero between $x = 1$ and $x = 2$, because it is continuous.

C.1.13 **Understand and apply the Extreme Value Theorem: If $f(x)$ is continuous over a closed interval, then f has a maximum and a minimum on the interval.**

Example: Decide if $t(x) = \tan x$ has a maximum value over the interval $[-\pi/4, \pi/4]$. What about the interval $[-\pi, \pi]$? Explain your answers.

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Standard 2

Differential Calculus

Students find derivatives of algebraic, trigonometric, logarithmic, and exponential functions. They find derivatives of sums, products, and quotients, and composite and inverse functions. They find derivatives of higher order and use logarithmic differentiation and the Mean Value Theorem.

C.2.1 **Understand the concept of derivative geometrically, numerically, and analytically, and interpret the derivative as a rate of change.**

Example: Find the derivative of $f(x) = x^2$ at $x = 5$ by calculating values of $\frac{x^2 - 5^2}{x - 5}$ for x near 5. Use a diagram to explain what you are doing and what the result means.

C.2.2 **State, understand, and apply the definition of derivative.**

Example: Find $\lim_{x \to 5} \frac{x^2 - 5^2}{x - 5}$. What does the result tell you?

C.2.3 **Find the derivatives of functions, including algebraic, trigonometric, logarithmic, and exponential functions.**

Example: Find dy/dx for the function $y = x^5$.

C.2.4 **Find the derivatives of sums, products, and quotients.**

Example: Find the derivative of $x \cos x$.

C.2.5 **Find the derivatives of composite functions, using the chain rule.**

Example: Find $f'(x)$ for $f(x) = (x^2 + 2)^4$.

C.2.6 **Find the derivatives of implicitly-defined functions.**

Example: For $xy - x^2y^2 = 5$, find dy/dx at the point $(2, 3)$.

C.2.7 **Find derivatives of inverse functions.**

Example: Let $f(x) = 2x^3$ and $g = f^{-1}$. Find $g'(2)$.

C.2.8 **Find second derivatives and derivatives of higher order.**

Example: Find the second derivative of e^{5x}.

C.2.9 **Find derivatives using logarithmic differentiation.**

Example: Find dy/dx for $y = \sqrt{(x + 3)^3 (x - 7)}$.

C.2.10 **Understand and use the relationship between differentiability and continuity.**

Example: Is $f(x) = 1/x$ continuous at $x = 0$? Is $f(x)$ differentiable at $x = 0$? Explain your answers.

C.2.11 **Understand and apply the Mean Value Theorem.**

Example: For $f(x) = \sqrt{x}$ on the interval $[1, 9]$, find the value of c such that $\frac{f(9) - f(1)}{9 - 1} = f'(c)$.

Standard 3

Applications of Derivatives

Students find slopes and tangents, maximum and minimum points, and points of inflection. They solve optimization problems and find rates of change.

C.3.1 **Find the slope of a curve at a point, including points at which there are vertical tangents and no tangents.**

Example: Find the slope of the tangent to $y = x^3$ at the point $(2, 8)$.

C.3.2 **Find a tangent line to a curve at a point and a local linear approximation.**

Example: In the last example, find an equation of the tangent at $(2, 8)$.

C.3.3 **Decide where functions are decreasing and increasing. Understand the relationship between the increasing and decreasing behavior of f and the sign of f'.**

Example: Use values of the derivative to find where $f(x) = x^3 - 3x$ is decreasing.

C.3.4 **Find local and absolute maximum and minimum points.**

Example: In the last example, find the local maximum and minimum points of $f(x)$.

C.3.5 **Analyze curves, including the notions of monotonicity and concavity.**

Example: In the last example, for which values of x is $f(x)$ decreasing and for which values of x is $f(x)$ concave upward?

C.3.6 **Find points of inflection of functions. Understand the relationship between the concavity of f and the sign of f''. Understand points of inflection as places where concavity changes.**

Example: In the last example, find the points of inflection of $f(x)$ and where $f(x)$ is concave upward and concave downward.

C.3.7 **Use first and second derivatives to help sketch graphs. Compare the corresponding characteristics of the graphs of f, f', and f''.**

Example: Use the last examples to draw the graph of $f(x) = x^3 - 3x$.

C.3.8 **Use implicit differentiation to find the derivative of an inverse function.**

Example: Let $f(x) = 2x^3$ and $g = f^{-1}$. Find $g'(x)$ using implicit differentiation.

C.3.9 **Solve optimization problems.**

Example: You want to enclose a rectangular area of 5,000 m^2. Find the shortest length of fencing you can use.

C.3.10 **Find average and instantaneous rates of change. Understand the instantaneous rate of change as the limit of the average rate of change. Interpret a derivative as a rate of change in applications, including velocity, speed, and acceleration.**

Example: You are filling a bucket with water and the height H cm of the water after t seconds is given by $H(t) = (4t)^{2/3}$. How fast is the water rising 30 seconds after you start filling the bucket? Explain your answer.

C.3.11 **Find the velocity and acceleration of a particle moving in a straight line.**

Example: A bead on a wire moves so that, after t seconds, its distance s cm from the midpoint of the wire is given by $s = 5 \sin (t - \pi/4)$. Find its maximum velocity and where along the wire this occurs.

C.3.12 **Model rates of change, including related rates problems.**

Example: A boat is heading south at 10 mph. Another boat is heading west at 15 mph toward the same point. At these speeds, they will collide. Find the rate that the distance between them is decreasing 1 hour before they collide.

Standard 4

Integral Calculus

Students define integrals using Riemann Sums, use the Fundamental Theorem of Calculus to find integrals, and use basic properties of integrals. They integrate by substitution and find approximate integrals.

C.4.1 Use rectangle approximations to find approximate values of integrals.

Example: Find an approximate value for $\int_0^3 x^2dx$ using 6 rectangles of equal width under the graph of $f(x) = x^2$.

C.4.2 Calculate the values of Riemann Sums over equal subdivisions using left, right, and midpoint evaluation points.

Example: Find the value of the Riemann Sum over the interval $[0, 3]$ using 6 subintervals of equal width for $f(x) = x^2$ evaluated at the midpoint of each subinterval.

C.4.3 Interpret a definite integral as a limit of Riemann Sums.

Example: Find the values of the Riemann Sums over the interval $[0, 3]$ using 12, 24, etc., subintervals of equal width for $f(x) = x^2$ evaluated at the midpoint of each subinterval. Find the limit of the Riemann Sums.

C.4.4 Understand the Fundamental Theorem of Calculus: Interpret a definite integral of the rate of change of a quantity over an interval as the change of the quantity over the interval, that is $\int_a^b f'(x)dx = f(b) - f(a)$.

Example: Explain why $\int_4^5 2xdx = 5^2 - 4^2$.

C.4.5 Use the Fundamental Theorem of Calculus to evaluate definite and indefinite integrals and to represent particular antiderivatives. Perform analytical and graphical analysis of functions so defined.

Example: Using antiderivatives, find $\int_0^3 x^2dx$.

C.4.6 Understand and use these properties of definite integrals:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b k \bullet f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Example: Find $\int_0^3 5x^2dx$, given that $\int_0^3 x^2dx = 9$.

C.4.7 Understand and use integration by substitution (or change of variable) to find values of integrals.

Example: Find $\int_1^2 x^2(x^3 + 1)^4dx$.

C.4.8 Understand and use Riemann Sums, the Trapezoidal Rule, and technology to approximate definite integrals of functions represented algebraically, geometrically, and by tables of values.

Example: Use the Trapezoidal Rule with 6 subintervals over $[0, 3]$ for $f(x) = x^2$ to approximate the value of $\int_0^3 x^2dx$.

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Standard 5

Applications of Integration

Students find velocity functions and position functions from their derivatives, solve separable differential equations, and use definite integrals to find areas and volumes.

C.5.1 **Find specific antiderivatives using initial conditions, including finding velocity functions from acceleration functions, finding position functions from velocity functions, and applications to motion along a line.**

Example: A bead on a wire moves so that its velocity, after t seconds, is given by $v(t) = 3 \cos 3t$. Given that it starts 2 cm to the left of the midpoint of the wire, find its position after 5 seconds.

C.5.2 **Solve separable differential equations and use them in modeling.**

Example: The slope of the tangent to the curve $y = f(x)$ is given by $-x/y$. Find an equation of the curve $y = f(x)$.

C.5.3 **Solve differential equations of the form $y' = ky$ as applied to growth and decay problems.**

Example: The amount of a certain radioactive material was 10 kg a year ago. The amount is now 9 kg. When will it be reduced to 1 kg? Explain your answer.

C.5.4 **Use definite integrals to find the area between a curve and the x-axis, or between two curves.**

Example: Find the area bounded by $y = \sqrt{x}$, $x = 0$, and $x = 2$.

C.5.5 **Use definite integrals to find the average value of a function over a closed interval.**

Example: Find the average value of $y = \sqrt{x}$ over $[0, 2]$.

C.5.6 **Use definite integrals to find the volume of a solid with known cross-sectional area.**

Example: A cone with its vertex at the origin lies symmetrically along the x-axis. The base of the cone is at $x = 5$ and the base radius is 7. Use integration to find the volume of the cone.

C.5.7 **Apply integration to model and solve problems in physics, biology, economics, etc., using the integral as a rate of change to give accumulated change and using the method of setting up an approximating Riemann Sum and representing its limit as a definite integral.**

Example: Find the amount of work done by a variable force.

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